Directed waves in random media: An analytical calculation

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The propagation of directed scalar waves in $D+1$ dimensions in a strongly disordered medium is studied. We use the model first proposed by Saul, Kardar, and Read [Phys. Rev. A **45**, 8859 (1992)], where unitarity is guaranteed in each step. The beam positions $\langle \overline{\mathbf{x}^2} \rangle$ and $\langle \overline{\mathbf{x}}^2 \rangle$ characterize the transverse fluctuations of a directed wave front, where the overbar means an average over the wave profile for a given realization of randomness, and $\langle\ \rangle$ means a quenched average over all realizations. We introduce $G_q^{\mathbf{k}}(\mathbf{y})$ as the Laplace-transformed Green function of two free random walkers with center-of-mass momentum $\mathbf{k}$ and relative position $\mathbf{y}$. We calculate analytically the mean-square deviation of the beam center, $\langle \overline{\mathbf{x}}^2 \rangle$, as a function of time. The results show that, for large t, $\langle \overline{\mathbf{x}}^2 \rangle$ behaves as $(1/\sqrt{\pi})t^{1/2}-\frac{1}{4}+O(t^{-3/2})$ in $1+1$ dimensions and as $(\ln t+4\ln 2+\gamma)/4\pi+O(1/t)$ in $2+1$ dimensions and takes the finite value $1/2D[G_{q=1}^{\mathbf{k}=0}(0)-\sqrt{(27/4\pi)}\,t^{-1/2}\delta_{D,3}]+O(1/t)$ in $D+1$ dimensions where $D\geq 3$, γ being the Euler constant. We generalize these results to a twofold random walk with any probability-flux-conserving interaction. In all cases the leading term at large t depends solely on the finite value or leading singularity of $G_q^{\mathbf{k}=0}(0)$ at $q=1$.

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I. INTRODUCTION

Consider a scalar wave $\phi(\mathbf{x},z)$ propagating in a random medium; the static solution of $\phi(\mathbf{x},z)$ satisfies the following Helmholtz equation:

$$[\nabla^2+k^2n^2(\mathbf{x},z)]\phi(\mathbf{x},z)=0 \tag{1.1}$$

where $n(\mathbf{x},z)$ is a nonuniform index of refraction that describes the disorder of the host medium. Following Feng, Golubovic, and Zhang (FGZ) [1], we decompose $n^2(\mathbf{x},z)=n_0^2+\delta n^2(\mathbf{x},z)$, where n_0 is the disordered-averaged index of refraction, and $\delta n^2(\mathbf{x},z)$ contains local fluctuations due to randomly distributed scattering centers. Assume the fluctuation varies slowly in the transverse directions (subspace $\mathbf{x}$), then it will favor a coherent propagation along the z direction. For such a case, we can set $\phi(\mathbf{x},z)=\Phi(\mathbf{x},z)e^{ikn_0z}$. We require $\delta n^2 \ll n^2$ and $\partial_z \delta n^2 \ll kn_0\delta n^2$ so that each scattering event causes only a small transverse momentum change, and that the term $\partial^2\Phi/\partial z^2$ (effect of back scattering) can be neglected. Now a Schrödinger equation is obtained by the changing of variables $z\leftrightarrow t$:

$$i\frac{\partial\Phi}{\partial t}=[-\mu\nabla_{\mathbf{x}}^2+V(\mathbf{x},t)]\Phi\ , \tag{1.2}$$

where $\mu=(2kn_0)^{-1}$ and $V=-k\delta n^2/2n_0$ [2]. Since both μ and V are real, the norm of the wave function is preserved. For a wave initially (at $t=0$) localized at $\mathbf{x}=0$, the solution is given by the Feynman path-integral formula

$$\Phi(\mathbf{x},t)=\int_{(0,0)}^{(\mathbf{x},t)}\mathcal{D}[\mathbf{x}(\tau)]\exp\left\{i\int_0^t d\tau\left[\frac{1}{2\mu}\left[\frac{d\mathbf{x}}{d\tau}\right]^2-V(\mathbf{x}(\tau),\tau)\right]\right\}\ , \tag{1.3}$$

where $\mathbf{x}(\tau)$ describes a path in D dimensions. We use an overbar to indicate an average with weight $|\Phi(x,t)|^2$ for a given realization, and $\langle\ \rangle$ to indicate the quenched average over all realizations of randomness. Approximately, $\langle \overline{\mathbf{x}}^2 \rangle$ describes the wandering of the beam center, while $\langle \overline{\mathbf{x}^2}-\overline{\mathbf{x}}^2 \rangle$ gives a measure of the beam width.

Equation (1.3) is to be discretized on a $(D+1)$-dimensional lattice Λ. To describe the lattice, consider a random walk in discrete time on a D-dimensional simple hypercubic spatial lattice. In each time step the walker moves one unit in space. Now visualize the $(D+1)$-dimensional lattice Λ containing all world lines of such walks. [For $D=1$, Λ is a square lattice rotated 45°; for $D=2$, Λ is a body-centered-cubic. These two cases are described by Saul, Kavdar, and Read (SKR).]

The wave function takes its values on links of the lattice Λ. We use $\Phi_{\mathbf{j}}(\mathbf{x},t)$ to refer to the amplitude arriving at the site $(\mathbf{x},t)$ from the $\mathbf{j}$ direction, where $\mathbf{j}$ represents a unit vector among the $2D$ possible unit vectors in D dimensions (i.e., $\mathbf{j}\in\{\pm\hat{\mathbf{e}}_1,\pm\hat{\mathbf{e}}_2,\ldots\pm\hat{\mathbf{e}}_D\}$). At $t=0$, the wave function is localized at the origin, with $\Phi_{\mathbf{j}}(0,0)=1/\sqrt{2D}$ for all $\mathbf{j}$. We then have to assign a complex-valued amplitude to each trajectory of arbitrary length t emanating from the origin; the wave function will be the sum of the amplitude of all trajectories arriving at a given link. Therefore we assign a $2D \times 2D$ unitary matrix $S(\mathbf{x},t)$ to each site on the lattice. The values of the wave function at time $t+1$ are then given from the following equation:

$$\begin{bmatrix} \Phi_1(\mathbf{x}-\mathbf{1}) \\ \Phi_2(\mathbf{x}-\mathbf{2}) \\ \vdots \\ \Phi_{\mathbf{j}}(\mathbf{x}-\mathbf{j}) \\ \vdots \end{bmatrix}_{t+1} = \begin{bmatrix} S_{1,1}(\mathbf{x}) & \cdots & S_{1,2D}(\mathbf{x}) \\ S_{2,1}(\mathbf{x}) & \cdots & S_{2,2D}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ S_{2D,1}(\mathbf{x}) & \cdots & S_{2D,2D}(\mathbf{x}) \end{bmatrix}_t \begin{bmatrix} \Phi_1(\mathbf{x}) \\ \Phi_2(\mathbf{x}) \\ \vdots \\ \Phi_{\mathbf{j}}(\mathbf{x}) \\ \vdots \end{bmatrix}_t . \quad (1.4)$$

To simulate the effect of a random potential, following SKR the S matrices are picked randomly from the fundamental representation of the $U(2D)$ group. One is interested in the second moment of the average flux,

$$\langle \overline{\mathbf{x}^2} \rangle = \sum_{\mathbf{x}} \langle P(\mathbf{x},t) \rangle \mathbf{x}^2 , \quad (1.5)$$

and in that of the distribution of the beam center,

$$\langle \overline{\mathbf{x}}^2 \rangle = \sum_{\mathbf{x}_1,\mathbf{x}_2} \langle P(\mathbf{x}_1,t) P(\mathbf{x}_2,t) \rangle \mathbf{x}_1 \cdot \mathbf{x}_2 . \quad (1.6)$$

Here, $P(\mathbf{x},t)$ is the probability distribution function on the lattice at time t, defined by

$$P(\mathbf{x},t) = \sum_{\mathbf{j}} |\Phi_{\mathbf{j}}(\mathbf{x},t)|^2 . \quad (1.7)$$

Note that $\sum_{\mathbf{x}} P(\mathbf{x},t)=1$ is guaranteed from the initial condition and unitarity. This is why Eqs. (1.6) and (1.5) are not divided by $\sum_{\mathbf{x}} P(\mathbf{x},t)$. The average $\langle\ \rangle$ in Eqs. (1.6) and (1.5) should be performed over a distribution of the S matrices that closely resembles the corresponding distribution for V in the continuum problem. However, we consider only the strong-scattering limit, where each matrix $S(x,t)$ is an *independently* chosen, random element of the group U($2D$). According to SKR, the results are expected to be valid for the continuous Eq. (1.1) over a range of length scales $d \ll z \ll \xi$, where d is a length over which the phase change caused by randomness is around 2π, and ξ is the length scale for the decay of intensity and breakdown of unitarity. In such a strong-scattering limit, the effect of $S(\mathbf{x},t)$ is therefore to redistribute the incident probability flux $P(\mathbf{x},t)$ at random in each possible direction. On the average, the flux is scattered symmetrically so that the average of $P(\mathbf{x},t)$ describes the probability distribution, in space, of a classical random walk; therefore $\langle \overline{\mathbf{x}^2} \rangle \propto t$. SKR drew attention to the quantity $\langle \overline{\mathbf{x}}^2 \rangle$, which is difficult to calculate because the correlation function $\langle P(\mathbf{x}_1,t) P(\mathbf{x}_2,t) \rangle$ does not have a simple form. We will show explicitly in Sec. II how to calculate this quantity.

II. FORMULATION AND CALCULATION

As pointed out by SKR, the quantity we are interested in depends only on $P(\mathbf{x},t)$; we therefore study its evolution. Consider the scattering event shown in Fig. 1. The probability flux is locally conserved. Define

$$g_{\mathbf{i}}(\mathbf{x},t) = \frac{|\Phi_{-\mathbf{i}}(\mathbf{x}+\mathbf{i},t+1)|^2}{\sum_{\mathbf{j}} |\Phi_{\mathbf{j}}(\mathbf{x},t)|^2} , \quad (2.1)$$

where $|\Phi_{-\mathbf{i}}|^2$ represents the outgoing probability flux into the $\mathbf{i}$ direction, and $|\Phi_{\mathbf{j}}|^2$ represents the incoming probability flux from the $\mathbf{j}$ direction. In view of (1.7) and unitarity, we have the following recursion relation:

$$P(\mathbf{x},t) = \sum_{\mathbf{j}} g_{\mathbf{j}}(\mathbf{x}-\mathbf{j},t-1) P(\mathbf{x}-\mathbf{j},t-1) . \quad (2.2)$$

To evaluate (1.6), we must study the evolution of the disorder-averaged correlation

$$\psi(\mathbf{x}_1,\mathbf{x}_2,t) \equiv \langle P(\mathbf{x}_1,t) P(\mathbf{x}_2,t) \rangle . \quad (2.3)$$

From (2.2) we obtain the recursion

$$\psi(\mathbf{x}_1,\mathbf{x}_2,t) = \sum_{\mathbf{j}_1,\mathbf{j}_2} \langle g_{\mathbf{j}_1}(\mathbf{x}_1-\mathbf{j}_1,t-1) g_{\mathbf{j}_2}(\mathbf{x}_2-\mathbf{j}_2,t-1) \rangle \times \psi(\mathbf{x}_1-\mathbf{j}_1,\mathbf{x}_2-\mathbf{j}_2,t-1) . \quad (2.4)$$

We must now study the quantities $\langle g_{\mathbf{i}}(\mathbf{x},t) g_{\mathbf{j}}(\mathbf{y},t) \rangle$. Strictly, the strong-scattering limit means that the matrix $S(\mathbf{x},t)$ is distributed uniformly with respect to the group measure of U($2D$). One then obviously has

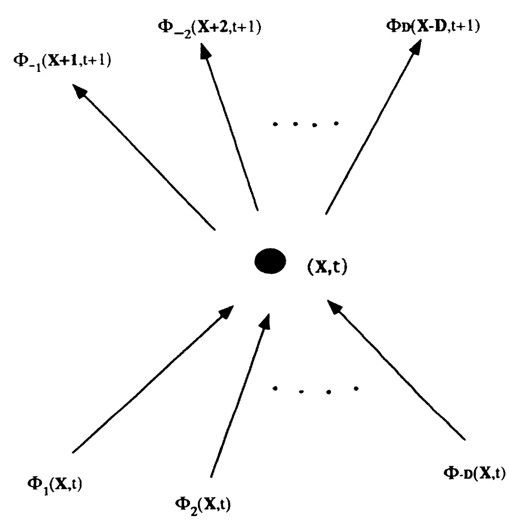

FIG. 1. Typical scattering event. The big dot represents the lattice point $(\mathbf{x},t)$. In the upper part of the figure, $\mathbf{i}$ runs from $-\mathbf{1}$ to $-\mathbf{D}$ and then from $\mathbf{1}$ to $\mathbf{D}$, while $|\Phi_{-\mathbf{i}}|^2$ represents the outgoing probability flux *into* the $\mathbf{i}$ direction. In the lower part of this figure, $\mathbf{j}$ runs from $\mathbf{1}$ to $\mathbf{D}$ and then from $-\mathbf{1}$ to $-\mathbf{D}$, while $|\Phi_{\mathbf{j}}|^2$ represents the incoming probability flux *from* the $\mathbf{j}$ direction.

$$\langle g_i(\mathbf{x},t)\rangle=\frac{1}{2D}\ ,\quad \langle g_i(\mathbf{x},t)g_j(\mathbf{y},t)\rangle_{\mathbf{x}\neq\mathbf{y}}=\left[\frac{1}{2D}\right]^2\ , \tag{2.5}$$

and, after some calculation,

$$\langle g_i^2(\mathbf{x},t)\rangle=2\langle g_i(\mathbf{x},t)g_j(\mathbf{x},t)\rangle_{i\neq j} =\frac{1}{D(2D+1)}\equiv\alpha^{\text{u.a.}}\ . \tag{2.6}$$

[We use the superscript u.a. to denote the uniform average over the group U(2D).]

However, we shall consider a more general stochastic problem [3] in which it is still true that the outgoing direction of the random trajectory is uninfluenced by its incoming direction, but the probability flux need not be derived from a complex amplitude as in (1.7). Thus we still assume (2.2)–(2.5) but replace (2.6) by

$$\langle g_i^2(\mathbf{x},t)\rangle=\alpha\ ,\qquad \langle g_i(\mathbf{x},t)g_j(\mathbf{x},t)\rangle_{i\neq j}=\frac{1}{2D-1}\left[\frac{1}{2D}-\alpha\right]\ , \tag{2.7}$$

as required by conservation of probability, where α is a real number constrained only by the inequality

$$\left[\frac{1}{2D}\right]^2=\langle g_i\rangle^2\le\langle g_i^2\rangle=\alpha\le\langle g_i\rangle=\frac{1}{2D}\ , \tag{2.8}$$

resulting from the fact that $g_i\le 1$ and from the Schwarz inequality. Note that (2.8) and the conservation of probability are satisfied by putting $\alpha=\alpha^{\text{u.a.}}$, as given by (2.6), into (2.7). Equation (2.4) now reads as follows:

$$\psi(\mathbf{x}_1,\mathbf{x}_2,t+1)=\frac{1}{4D^2}\sum_{\mathbf{j}_1,\mathbf{j}_2}\psi(\mathbf{x}_1-\mathbf{j}_1,\mathbf{x}_2-\mathbf{j}_2,t)\times[1+\Delta(\mathbf{x}_1,\mathbf{x}_2,\mathbf{j}_1,\mathbf{j}_2)]\ ,\qquad \psi(\mathbf{x}_1,\mathbf{x}_2,0)=\delta^D_{\mathbf{x}_1,0}\delta^D_{\mathbf{x}_2,0}\ , \tag{2.9}$$

where

$$\Delta(\mathbf{x}_1,\mathbf{x}_2,\mathbf{j}_1,\mathbf{j}_2)=(4D^2\alpha-1)\left[\frac{2D}{2D-1}\delta^D_{\mathbf{x}_1,\mathbf{x}_2}\delta^D_{\mathbf{x}_1-\mathbf{j}_1,\mathbf{x}_2-\mathbf{j}_2}-\frac{1}{2D-1}\delta^D_{\mathbf{x}_1-\mathbf{j}_1,\mathbf{x}_2-\mathbf{j}_2}\right]\ .$$

This recursion has already been studied by the authors [4] in the context of the exact solution of the two-replica model of the directed polymer problem. We introduce the generating functions:

$$\psi_q(\mathbf{x}_1,\mathbf{x}_2)\equiv\sum_{t=0}^{\infty}q^t\psi(\mathbf{x}_1,\mathbf{x}_2,t)\ ,\qquad G_q(\mathbf{x}_1,\mathbf{x}_2)\equiv\sum_{t=0}^{\infty}q^tG(\mathbf{x}_1,\mathbf{x}_2,t)\ . \tag{2.10}$$

where G is the "free-particle" Green function which satisfies

$$G(\mathbf{x}_1,\mathbf{x}_2,t+1)=\frac{1}{4D^2}\sum_{\mathbf{j}_1}\sum_{\mathbf{j}_2}G(\mathbf{x}_1-\mathbf{j}_1,\mathbf{x}_2-\mathbf{j}_2,t)\ ,\qquad G(\mathbf{x}_1,\mathbf{x}_2,0)=\delta^D_{\mathbf{x}_1,0}\delta^D_{\mathbf{x}_2,0}\ . \tag{2.11}$$

From (2.9), we then have

$$\psi_q(\mathbf{x}_1,\mathbf{x}_2)=G_q(\mathbf{x}_1,\mathbf{x}_2)+\frac{q}{4D^2}\sum_{\mathbf{x}_1',\mathbf{x}_2'}\sum_{\mathbf{j}_1,\mathbf{j}_2}\psi_q(\mathbf{x}_1'-\mathbf{j}_1,\mathbf{x}_2'-\mathbf{j}_2)G_q(\mathbf{x}_1-\mathbf{x}_1',\mathbf{x}_2-\mathbf{x}_2')\cdot\Delta(\mathbf{x}_1',\mathbf{x}_2',\mathbf{j}_1,\mathbf{j}_2)\ . \tag{2.12}$$

It is now helpful to introduce the "relative position vector" $\mathbf{y}=(\mathbf{x}_1-\mathbf{x}_2)/2$ (with integer components) and the "total momentum" $\mathbf{k}$. We define

$$\psi_q^{\mathbf{k}}(\mathbf{y})\equiv\sum_{\mathbf{x}_1,\mathbf{x}_2}e^{-i\mathbf{k}(\mathbf{x}_1+\mathbf{x}_2)/2}\delta^D_{\mathbf{x}_1-\mathbf{x}_2,2\mathbf{y}}\psi_q(\mathbf{x}_1,\mathbf{x}_2)\ ,\qquad G_q^{\mathbf{k}}(\mathbf{y})\equiv\sum_{\mathbf{x}_1,\mathbf{x}_2}e^{-i\mathbf{k}(\mathbf{x}_1+\mathbf{x}_2)/2}\delta^D_{\mathbf{x}_1-\mathbf{x}_2,2\mathbf{y}}G_q(\mathbf{x}_1,\mathbf{x}_2)\ , \tag{2.13}$$

we then have

$$\psi_q^{\mathbf{k}}(\mathbf{y})=G_q^{\mathbf{k}}(\mathbf{y})+(4D^2\alpha-1)\frac{q}{4D^2}J_q^{\mathbf{k}}(\mathbf{y})\psi_q^{\mathbf{k}}(0)\ , \tag{2.14}$$

where

$$J_q^{\mathbf{k}}(\mathbf{y})=\frac{2D}{2D-1}\left[\sum_{\mathbf{i}}\cos(\mathbf{k}\cdot\mathbf{i})\right]G_q^{\mathbf{k}}(\mathbf{y})-\frac{1}{2D-1}\sum_{\mathbf{i},\mathbf{j}}\exp\left[-i\mathbf{k}\cdot\frac{\mathbf{i}+\mathbf{j}}{2}\right]G_q^{\mathbf{k}}\left[\mathbf{y}+\frac{\mathbf{i}-\mathbf{j}}{2}\right]\ . \tag{2.15}$$

Now make one further Fourier transformation,

$$\psi_q^{\mathbf{k},\mathbf{p}} \equiv \sum_{\mathbf{y}} e^{-i\mathbf{p}\cdot\mathbf{y}} \psi_q^{\mathbf{k}}(\mathbf{y}) = G_q^{\mathbf{k},\mathbf{p}} + (4D^2\alpha - 1)\frac{q}{4D^2}\psi_q^{\mathbf{k}}(0) J_q^{\mathbf{k},\mathbf{p}} , \tag{2.16}$$

where

$$J_q^{\mathbf{k},\mathbf{p}} = \sum_{\mathbf{y}} e^{-i\mathbf{p}\cdot\mathbf{y}} J_q^{\mathbf{k}}(\mathbf{y}) = G_q^{\mathbf{k},\mathbf{p}} \left\{ \frac{2D}{2D-1} \sum_{\mathbf{i}} \cos(\mathbf{k}\cdot\mathbf{i}) - \frac{1}{2D-1} \left[\sum_{\mathbf{i}} \cos\frac{\mathbf{k}+\mathbf{p}}{2}\cdot\mathbf{i} \right] \left[\sum_{\mathbf{j}} \cos\frac{\mathbf{k}-\mathbf{p}}{2}\cdot\mathbf{j} \right] \right\} , \tag{2.17}$$

from which [using the double Fourier transform of (2.11)]

$$J_q^{\mathbf{k}}(0) = \frac{2D}{2D-1} \left[\sum_{\mathbf{i}} \cos(\mathbf{k}\cdot\mathbf{i}) \right] G_q^{\mathbf{k}}(0) + \frac{4D^2}{q(2D-1)} [1 - G_q^{\mathbf{k}}(0)] ; \tag{2.18}$$

furthermore, setting $\mathbf{y}=0$ in (2.14),

$$\psi_q^{\mathbf{k}}(0) = \frac{G_q^{\mathbf{k}}(0)}{1 - [q(4D^2\alpha - 1) J_q^{\mathbf{k}}(0)/4D^2]} . \tag{2.19}$$

Combining Eqs. (2.17) and (2.19), we see that the final answer for Eq. (2.16) depends only on the Green function $G_q^{\mathbf{k},\mathbf{p}}$, which characterizes the free random walk. Up to this point we have followed [4]. We are now ready to calculate the quantity

$$\sum_{\mathbf{x}_1,\mathbf{x}_2} \psi_q(\mathbf{x}_1,\mathbf{x}_2)\mathbf{x}_1\cdot\mathbf{x}_2 = \{(\nabla_{\mathbf{p}}^2 - \nabla_{\mathbf{k}}^2)\psi_q^{\mathbf{k},\mathbf{p}}\}|_{\mathbf{k}=\mathbf{p}=0} = (4D^2\alpha - 1)\frac{q}{4D^2}\psi_q^{\mathbf{k}=0}(0)\{(\nabla_{\mathbf{p}}^2 - \nabla_{\mathbf{k}}^2) J_q^{\mathbf{k},\mathbf{p}}\}|_{\mathbf{k}=\mathbf{p}=0}$$

by (2.16) and simple symmetries; note that $G_q^{\mathbf{k},\mathbf{p}}$ is symmetric in $\mathbf{k},\mathbf{p}$, and that $J_q^{0,0}=0$ by (2.17). Further, from (2.17) and symmetries we have

$$(\nabla_{\mathbf{p}}^2 - \nabla_{\mathbf{k}}^2) J_q^{\mathbf{k},\mathbf{p}}|_{\mathbf{k}=\mathbf{p}=0} = G_q^{0,0}\frac{2D}{2D-1}(-\nabla_k^2)\left[\sum_{\mathbf{i}} \cos(\mathbf{k}\cdot\mathbf{i}) \right]\Bigg|_{\mathbf{k}=0} = \frac{1}{1-q}\frac{4D^2}{2D-1}$$

since $G_q^{0,0} = 1/(1-q)$ by doubly Fourier transforming (2.10). Thus

$$\sum_{\mathbf{x}_1,\mathbf{x}_2} \psi_q(\mathbf{x}_1,\mathbf{x}_2)\mathbf{x}_1\cdot\mathbf{x}_2 = \frac{4D^2\alpha - 1}{2D-1}\frac{q}{1-q}\psi_q^{\mathbf{k}=0}(0) = \frac{q}{1-q}\frac{G_D(q)}{a + (1-q)G_D(q)} \tag{2.20}$$

by (2.19), where

$$a = \frac{2D-1}{4D^2\alpha - 1} - 1 = \frac{(2D)^{-1} - \alpha}{\alpha - (4D^2)^{-1}}$$

and

$$G_D(q) \equiv G_q^{\mathbf{k}=0}(0) = \left[\frac{1}{2\pi} \right]^D \int_0^{2\pi} G_q^{\mathbf{k}=0,\mathbf{p}} d^D\mathbf{p} = \left[\frac{1}{2\pi} \right]^D \int_0^{2\pi} \frac{d^D\mathbf{p}}{1 - \frac{q}{4D^2}\left[\sum_{\mathbf{j}} \cos \mathbf{p}\cdot\mathbf{j}/2 \right]^2} . \tag{2.21}$$

Note that $0 \le a \le \infty$ by (2.8), and $a^{\text{u.a.}} = 2D$ by (2.6). But the case $a=0$ is trivial: then $\alpha = 1/2D$, which means

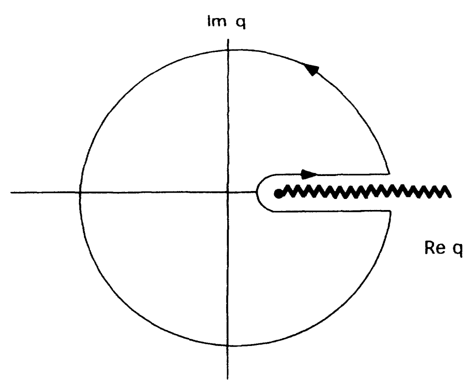

FIG. 2. Contour in the q plane for Eq. (2.22). The original contour is an infinitesimal counterclockwise circle about the origin. Only the deformed contour is shown in this figure.

that two paths always stick together and $\langle \bar{\mathbf{x}}^2\rangle = \langle \overline{\mathbf{x}^2}\rangle$. *Hence we need only consider $a>0$ in the following calculation.*

We now have

$$\langle \bar{\mathbf{x}}^2\rangle_t = \sum_{\mathbf{x}_1,\mathbf{x}_2} \psi(\mathbf{x}_1,\mathbf{x}_2,t)\mathbf{x}_1\cdot\mathbf{x}_2 = \frac{1}{2\pi i}\oint_{|q|=\epsilon}\frac{dq}{q^{t+1}}\frac{\frac{q}{1-q}G_D(q)}{a+(1-q)G_D(q)} , \quad (2.22)$$

where the contour is an infinitesimal counterclockwise circle about the origin. We extend $G_D(q)$ via (2.21) to the whole complex q plane. It can be shown that this extension is analytic except for a branch cut on the real axis with $q\ge 1$, and that the imaginary part of $\lim_{\delta\to 0}G_D(q+i\delta)$ near $q=1$ along the branch cut has leading term $[\Omega/2(2\pi)^{D-1}]D^{D/2}(q-1)^{(D/2)-1}$ (see Appendix A).

To extract the asymptotic ($t\to\infty$) behavior of the integral in (2.22), we deform the contour as shown in Fig. 2. It can be shown that $a+1(1-q)G_D(q)$ has no zeros within the entire region enclosed by our deformed contour (see Appendix B). We can see from (2.21) that $G_D(q)$ is finite for all q when $D\ge 3$; however, $G_D(q)$ diverges infraredly as $q\to 1$ when $D=1$ or 2.

For $D=1$, $G_1(q)=1/\sqrt{1-q}$, after a contour deformation (2.22) becomes

$$\begin{aligned}\langle \bar{\mathbf{x}}^2\rangle_t &= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\oint_{|q|=\epsilon}\frac{dq}{q^t}\frac{1}{a(1-q)^{3/2}+(1-q)^2}\\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\left\{\oint_{|1-q|=\epsilon}\frac{dq}{q^t}\frac{1}{a(1-q)^{3/2}+(1-q)^2}+\int_{1+\epsilon}^{\infty}\left[\frac{1}{ia(q-1)^{3/2}+(q-1)^2}-\frac{1}{-ia(q-1)^{3/2}+(q-1)^2}\right]\frac{dq}{q^t}\right\}\\ &= \lim_{\epsilon\to 0}\left\{\frac{2}{a\pi\epsilon^{1/2}}-\frac{1}{a^2}-\frac{1}{\pi}\int_{1+\epsilon}^{\infty}\frac{dq}{q^t}\frac{a(q-1)^{3/2}}{a^2(q-1)^3+(q-1)^4}\right\}\\ &= \frac{1}{\pi}\int_1^{\infty}dq\left[\frac{1}{a(q-1)^{3/2}}-\frac{1}{q^t}\frac{a(q-1)^{3/2}}{a^2(q-1)^3+(q-1)^4}\right]-\frac{1}{a^2} .\end{aligned}$$

Note that the above expression is a monotonic increasing function of t, and it is easy to show from it that $\langle \bar{\mathbf{x}}^2\rangle|_{t=0}=0$. The positive definiteness of $\langle \overline{\mathbf{x}^2}\rangle$ is therefore explicit. To extract the large-t behavior, we change variable by setting $u^2\equiv q-1$; then the above equation becomes

$$\begin{aligned}\langle \bar{\mathbf{x}}^2\rangle_t &= \frac{2}{\pi a}\int_0^{\infty}du\left\{\frac{1}{u^2}-\frac{1}{(1+u^2)^t}\left[\frac{1}{u^2}-\frac{1/a^2}{1+(u^2/a^2)}\right]\right\}-\frac{1}{a^2}\\ &= \frac{2}{\pi a}\left\{\frac{1}{u}\left[\frac{1}{(1+u^2)^t}-1\right]\Bigg|_0^{\infty}+\int_0^{\infty}du\left[\frac{2t}{(1+u^2)^{t+1}}+\frac{1/a^2}{(1+u^2)^t[1+(u^2/a^2)]}\right]\right\}-\frac{1}{a^2} .\end{aligned}$$

The first term is zero upon taking the upper and lower limits; the second term is evaluated by Gaussian approximation. Finally we have

$$\begin{aligned}\langle \bar{\mathbf{x}}^2\rangle_t &= \frac{2t^{1/2}}{a\sqrt{\pi}}\left[1-\frac{1}{8t}\right]-\frac{1}{a^2}+\frac{1}{a^2\sqrt{\pi t}}+O(t^{-3/2})\\ &= \frac{2t^{1/2}}{a\sqrt{\pi}}-\frac{1}{a^2}+\frac{1}{a\sqrt{\pi}}\left[\frac{1}{a^2}-\frac{1}{4}\right]t^{-1/2}+O(t^{-3/2})\\ &= \frac{t^{1/2}}{\sqrt{\pi}}-\tfrac{1}{4}+O(t^{-3/2})\ \text{ if } a=a^{\text{u.a.}}=2 .\end{aligned}\qquad (2.23)$$

For $D=2$, $G_2(q)=(2/\pi)K(\sqrt{q})$, where $K(\sqrt{q})=\int_0^{\pi/2}(d\theta/\sqrt{1-q\sin^2\theta})$ is the complete elliptic integral of the first kind. After a contour deformation, and taking the expansion of $G_2(q)$ around $q=1$ as

$$G_2(q)=\frac{1}{\pi}\left\{\ln\frac{16}{1-q}+O\left[(1-q)\ln\left[\frac{1}{1-q}\right]\right]\right\} ,$$

we have

$$
\begin{aligned}
\langle \bar{\mathbf{x}}^2 \rangle_t &= \frac{1}{2\pi i} \oint_{|q|=\epsilon} \frac{dq}{q^t(1-q)} \frac{G_2(q)}{a+(1-q)G_2(q)} \\
&= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \left\{ \oint_{|1-q|=\epsilon,\arg(1-q):\ \pi\to-\pi} \frac{dq}{q^t(1-q)} \frac{G_2(q)}{a+(1-q)G_2(q)} \right. \\
&\qquad \left. + \int_{1+\epsilon}^{\infty} \frac{dx}{x^t(1-x)} \left[\frac{G_2(x+i\delta)}{a+(1-x)G_2(x+i\delta)} - \frac{G_2(x-i\delta)}{a+(1-x)G_2(x-i\delta)} \right] \right\} \\
&= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \left\{ \frac{2i}{a} \ln\frac{16}{\epsilon} + \int_{1+\epsilon}^{\infty} \frac{x^{-t}dx}{(1-x)} \left[\frac{G_2(x+i\delta)}{a+(1-x)G_2(x+i\delta)} - (x+i\delta \to x-i\delta) \right] \right\} \\
&= \frac{1}{\pi} \left\{ \frac{4\ln 2}{a} + \int_1^{\infty} \frac{dx}{ax} \left[\frac{1}{x-1} - \frac{x^{-t}}{x-1} \right] \right. \\
&\qquad \left. + \int_1^{\infty} \frac{x^{-t-1}dx}{x-1} \left[\frac{1}{a} - \frac{x}{2i} \left[\frac{G_2(x+i\delta)}{a+(1-x)G_2(x+i\delta)} - \frac{G_2(x-i\delta)}{a+(1-x)G_2(x-i\delta)} \right] \right] \right\} ,
\end{aligned}
\tag{2.24}
$$

where $\ln(1/\epsilon)$ has been represented as $\int_{1+\epsilon}^{\infty} dx/[x(x-1)]$ and a term added and subtracted, after which the limit $\epsilon \to 0$ was taken. We will show in Appendix C that the last term of the above expression is bounded by $O(1/t)$. The second term is evaluated by writing it as

$$
\begin{aligned}
\frac{1}{a}\int_1^{\infty} \frac{x^{-(t+1)}dx}{x-1}(x^t-1) &= \frac{1}{a}\sum_{k=1}^{t}\frac{1}{k} \\
&= \frac{1}{a}[\gamma + \ln t + O(t^{-1})] .
\end{aligned}
$$

Thus

$$
\langle \bar{\mathbf{x}}^2 \rangle_t = \frac{1}{a\pi}(\ln t + 4\ln 2 + \gamma) + O\left[\frac{1}{t}\right] . \tag{2.25}
$$

For uniform averaging, $a = a^{\text{u.a.}} = 4$.

For $D \geq 3$, write $G_D(q+i\delta) = R_D(q) + iI_D(q)$. The integral along the branch cut can now be written as

$$
\frac{1}{\pi}\int_{1+\epsilon}^{\infty} \frac{aI_D(q)/(1-q)}{[a+(1-q)R_D]^2 + [(1-q)I_D]^2} \frac{dq}{q^t} .
$$

We know from Appendix A that $I_D(q) \sim (q-1)^{(D/2)-1}$, therefore when $D \geq 4$ the integral is bounded by $O(1/t)$; however, when $D = 3$, there is a subleading term given by taking the asymptotic expressions of the G_D and I_D near $q = 1$ (see Appendix A).

$$
\begin{aligned}
\frac{1}{\pi}\int_1^{\infty} \frac{-1}{2a} 3^{3/2}(q-1)^{-1/2} q^{-t} dq & \\
&= -\frac{\sqrt{27\pi}}{2\pi a} t^{-1/2} + O(t^{-3/2}) .
\end{aligned}
$$

The leading term is the integral around the branch point, which is finite and can be read off from Eq. (2.22), and from which the final answer reads

$$
\langle \bar{\mathbf{x}}^2 \rangle_t = \frac{1}{a}\left[G_D(q=1) - \left[\frac{27}{4\pi}\right]^{1/2} t^{-1/2}\delta_{D,3} \right] + O(1/t) . \tag{2.26}
$$

III. ANALYSIS AND SUMMARY

We have found the beam center spread as a function of time analytically for a directed scalar wave in a random medium of arbitrary dimension. The coefficients of successive powers of $t^{-1/2}$ turned out to be quite complicated. They are (for arbitrary dimension) polynomials in $1/a$, where $a = [2D-1]/[4D^2\alpha - 1] - 1 = [(2D)^{-1} - \alpha]/[\alpha - (4D^2)^{-1}]$. Compared with the analysis in SKR, where the expansion parameter is assumed to be $t^{-D}[4\alpha - (1/D^2)]$, we see some discrepancies. Thus our Eq. (2.23) does not agree with the form of Eq. (3.3) of SKR.

Our results can be verified independently in two extreme cases. In the noninteracting case, $\langle \bar{\mathbf{x}}^2 \rangle_t$ is expected to be zero. In this case, $a \to \infty$, and therefore our result also gives zero. In the strong attraction case (where the two paths almost always stick together), $\langle \bar{\mathbf{x}}^2 \rangle_t$ is expected to diverge at large t. In this case, $\alpha \to (2D)^{-1}+$ and therefore $a \to 0$, giving a divergent value of $\langle \bar{x}^2 \rangle_t$ as expected.

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APPENDIX A

Let q be real; consider the quantity

$$
I_D(q) \equiv \lim_{\delta\to\infty} \frac{1}{2i}[G_D(q+i\delta) - G_D(q-i\delta)] . \tag{A1}
$$

From equation (2.21), we have

$$
I_D(q) = \pi \left[\frac{1}{2\pi}\right]^D \int_0^{2\pi} \delta\left[1 - \frac{q}{D^2}\left[\sum_i \cos\frac{p_i}{2}\right]^2\right] \prod_i dp_i . \tag{A2}
$$

From this expression we know that $I_D(q)$, the imaginary part of $G_D(q+i\delta)$, has a nonvanishing value when $q>1$, and is zero when $q\le 1$.

To extract the leading power in $q-1$ when $q\sim 1$, we expand (A2) around $q=1$. Note that for $q\sim 1$ only the regions where $p_i\sim 0$ or $p_i\sim 2\pi$ contribute. We also know by the symmetry $p_i\leftrightarrow 2\pi-p_i$ that these two regions make the same contribution. We therefore have

$$I_D(q)|_{q\sim 1}\simeq\left[\frac{1}{2\pi}\right]^{D-1}\int_{p_i\ge 0}\delta\left[\left[\sum_{i=1}^{D}p_i^2\right]\Big/4D-(q-1)\right]\prod dp_i=\frac{\Omega D^{D/2}}{2(2\pi)^{D-1}}(q-1)^{(D/2)-1}, \tag{A3}$$

where Ω is the solid angle in D spatial dimensions.

APPENDIX B

Taking the original expression of $G_D(q)$, we consider the following quantity, denoting by q^* the complex conjugate of q:

$$(1-q)G_D(q)=\left[\frac{1}{2\pi}\right]^D\int_0^{2\pi}d^D\mathbf{p}\frac{(1-q)\left[1-\frac{q^*}{4D^2}\left[\sum_{\mathbf{j}}\cos\frac{\mathbf{p}\cdot\mathbf{j}}{2}\right]^2\right]}{\left[1-\frac{q}{4D^2}\left[\sum_{\mathbf{j}}\cos\frac{\mathbf{p}\cdot\mathbf{j}}{2}\right]^2\right]\left[1-\frac{q^*}{4D^2}\left[\sum_{\mathbf{j}}\cos\frac{\mathbf{p}\cdot\mathbf{j}}{2}\right]^2\right]}. \tag{B1}$$

Taking the imaginary part, we have

$$\mathrm{Im}\{(1-q)G_D(q)\}=\left[\frac{1}{2\pi}\right]^D\int_0^{2\pi}d^D\mathbf{p}\frac{(\mathrm{Im}q)\left[1-\frac{1}{4D^2}\left[\sum_{\mathbf{j}}\cos\frac{\mathbf{p}\cdot\mathbf{j}}{2}\right]^2\right]}{\left[1-\frac{q}{4D^2}\left[\sum_{\mathbf{j}}\cos\frac{\mathbf{p}\cdot\mathbf{j}}{2}\right]^2\right]\left[1-\frac{q^*}{4D^2}\left[\sum_{\mathbf{j}}\cos\frac{\mathbf{p}\cdot\mathbf{j}}{2}\right]^2\right]} \tag{B2}$$

which cannot vanish if $\mathrm{Im}q\neq 0$. Hence $a+(1-q)G_D(q)$ has no zeros off the real line. When q is real and <1, both $(1-q)$ and $G_D(q)$ are positive, and therefore $a+(1-q)G_D(q)$ is positive. Therefore $a+(1-q)G_D(q)$ never vanishes within the region surrounded by our deformed contour.

APPENDIX C

To show that the last term of Eq. (2.24) is bounded by $O(1/t)$, we use the fact [5] that for $|q|\ge 1$, with the convention that $-\pi\le\arg q<\pi$,

$$G_2(q)=\begin{cases}\frac{1}{\sqrt{q}}[G_2(1/q)+iG_2(1-1/q)] & \text{if } \mathrm{Im}q>0\\ \frac{1}{\sqrt{q}}[G_2(1/q)-iG_2(1-1/q)] & \text{if } \mathrm{Im}q<0\,.\end{cases}$$

We now make the variable change $y\equiv 1/x$ and define $G_2'(q)\equiv G_2(1-q)$. We then have

$$\int_1^\infty\frac{x^{-t-1}dx}{x-1}\left[\frac{1}{a}-\frac{x}{2i}\left[\frac{G_2(x+i\delta)}{a+(1-x)G_2(x+i\delta)}-\frac{G_2(x-i\delta)}{a+(1-x)G_2(x-i\delta)}\right]\right]$$
$$=\int_0^1 y^t dy\frac{1}{1-y}\left[\frac{1}{a}-\frac{ay^{1/2}G_2'(y)}{[ay^{1/2}+(y-1)G_2(y)]^2+[(y-1)G_2'(y)]^2}\right]. \tag{C1}$$

The function $G_2(y)$ is continuous and finite as long as $0\le y<1$, and diverges as $\ln(1-y)$ when y approaches 1; the function $G_2'(y)$ is finite as long as $0<y\le 1$, and diverges as $\ln y$ when y approaches 0. Using the above information, we note that the quantity

$$\frac{1}{1-y}\left[\frac{1}{a}-\frac{ay^{1/2}G_2'(y)}{[ay^{1/2}+(y-1)G_2(y)]^2+[(y-1)G_2'(y)]^2}\right]$$

as a function of y is continuous over the open interval (0,1), and stays finite as $y\to 0$ and $y\to 1$; this function therefore is bounded in the interval $0\le y\le 1$. The integral (C1) is therefore bounded by $O(1/t)$.

[1] S. Feng, L. Golubovic, and Y. C. Zhang, Phys. Rev. Lett. **65**, 1028 (1990).

[2] We have defined V with the sign opposite from that of Saul, Kardar, and Read (SKR). In addition, the signs of $\partial\Psi/\partial z$ in Eq. (1.2) and of $\gamma\nabla^2$ in Eq. (1.3) in SKR appear to be typographically incorrect.

[3] Saul, Kordar, and Read considered $\alpha^{\mathrm{u.a.}}$ [what they called $(\epsilon D^2+1)/4D^2$] to be undetermined. Since we have determined the exact value of $\alpha^{\mathrm{u.a.}}$, we have no direct physical motivation for considering any other value. [Nonuniform distributions of the scattering matrices over $U(2D)$ lead to a problem that cannot be treated in terms of $P(\mathbf{x},t)$ alone, even if one takes a modified α.] Nevertheless, we have treated α as a variable parameter in order to obtain exact answers for comparison with numerical calculations by SKR.

[4] R. Friedberg and Y. K. Yu, Phys. Rev. E 49, 4157 (1994).

[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (corrected and enlarged edition) (Academic, New York, 1980), p. 908.